Recitation Problems

Problem 1. *Let G, H be groups and let $\phi : G \to H$ be a homomorphism. Show that $\phi(G)$ is a subgroup of H (without using the subgroup criterion). If ϕ is one-to-one, show that $G \cong \phi(G)$.*

Problem 2. *Let $Z(G)$ be the center of the group G. Prove if $G/Z(G)$ is cyclic, the G is abelian. In other words, $G/Z(G) = \{e\}$.*

Problem 3. Prove or disprove the following*: Let H_1, H_2 be groups and define $G = H_1 \times H_2$. Then every subgroup of G is of the form $K_1 \times K_2$ where $K_1 \leq H_1$, $K_2 \leq H_2$*

Problem 4. *Consider the additive group $\mathbb{Q}/\mathbb{Z}$. Prove:*

1. *For each $n \in mathbbN$ there is an element of $\mathbb{Q}/\mathbb{Z}$ of order n.*
2. *There is a unique subgroup of $\mathbb{Q}/\mathbb{Z}$ of order n for each $n \in \mathbb{N}$.*

Problem 5. *Let G be a group and let M be a maximal subgroup of G. That is, M is a proper subgroup of G and for any subgroup H of G, if $M \subsetneq H$, then $H = G$. Prove that if $|G : M| < \infty$ and M is normal in G, then $|G : M|$ is prime.*

Problem 6. *Let G be a finite simple group and let $k \in \mathbb{N}$ be the smallest positive integer such that $|G| \mid k!$. Prove that if $H \leq G$, then $|G : H| \geq k$.*

Problem 7. *Let G be an infinite simple group. If $H \leq G$, then $|G : H|$ is infinite as well.*

Problem 8. *Let G be a finite simple group with $|G| \geq 60$. Prove that G has no subgroups of index less than 5.*

Problem 9. *Let G be a finite simple group with a subgroup H of prime index p. Show that p must be the largest prime dividing the order of G.*

Problem 10. *Let p be a prime and let $Z = \{z \in \mathbb{C} | z^{p^n} = 1$ for some $n \in \mathbb{Z}^+\}$ (so Z is the multiplicative group of all p-power roots of unity in $\mathbb{C}$). For each $k \in \mathbb{Z}^+$ let $H_k = \{z \in Z | z^{p^k} = 1\}$ (the group of p^kth roots of unity). Prove the following:*

(a) *$H_k \leq H_m$ if and only if $k \leq m$.*

(b) *H_k is cyclic for all k (assume that for any $n \in \mathbb{Z}^+$, $\{e^{2\pi it/n} | t = 1, 2, \ldots, n-1\}$ is the set of all n^{th} roots of 1 in $\mathbb{C}$)*

(c) *every proper subgroup of Z equals H_k for some $k \in \mathbb{Z}^+$ (in particular, every proper subgroup of Z is finite and cyclic)*

(d) *Z is not finitely generated*

Problem 11. *Let G be a finite group and let $\pi : G \to S_G$ be the left regular representation. Prove that if x is an element of G of order n and $|G| = mn$, then $\pi(x)$ is a product of m n-cycles. Deduce that $\pi(x)$ is an odd permutation if and only if $|x|$ is even and $\dfrac{|G|}{|x|}$ is odd.*

Problem 12. *Let G be a group of order 160. Show that G is not simple. Furthermore, show that G must have a normal 2-subgroup.*

Problem 13. *Suppose G is a simple group of order $168 = 2^3 \cdot 3 \cdot 7$.*

(a) *Prove G is not abelian.*

(b) *Determine the number of elements of order 7 in G.*

(c) *Prove that G has no proper subgroups of index less than 7 (hint: use the homomorphism to S_m given by a subgroup of index m).*

(d) *If H_2 is a Sylow 2-subgroup of G and H_7 is a Sylow 7-subgroup of G, then G is generated by $H_2 \cup H_7$.*

(e) *Prove that the number of Sylow 3-subgroups in G is a multiple of 7.*

Problem 14. *Let p be an odd prime. Show that $D_p \cong P \rtimes Q$ where $P \in Syl_p(D_p)$ and $Q \in Syl_2(D_p)$. Let r correspond to the "rotation" of the p-gon and let s correspond the the "reflection." Show that the operation on $P \rtimes Q$ corresponds to the operation of D_p. That is, $sr = r^{-1}s$ in D_p so we must have $(r, e) \bullet (e, s) = (e, s) \bullet (e, r^{-1})$ in $P \rtimes Q$.*

Problem 15. *Show that there is a nontrivial semi-direct product of C_4 and S_3. That is, show that there is a way to define an operation of $C_4 \rtimes S_3$ such that $C_4 \rtimes S_3 \neq C_4 \times S_3$. Another way to state the problem would be: show that we can define an operation on the set of ordered pairs (c, s) with $c \in C_4$ and $s \in S_3$ so that $C_4 \trianglelefteq C_4 \rtimes S_3$, but $S_3 \ntrianglelefteq C_4 \rtimes S_3$.*

Problem 16. *For any group G define the dual group of G (denoted $\widehat{G}$) to be the set of all homomorphisms from G into the multiplicative group of roots of unity in $\mathbb{C}$. Define a group operation in $\widehat{G}$ by pointwise multiplication of functions: if χ, ψ are homomorphisms from G into the group of roots of unity then $\chi\psi$ is the homomorphism given by $(\chi\psi)(g) = \chi(g)\psi(g)$ for all $g \in G$, where the latter multiplication takes place in $\mathbb{C}$.*

(a) *Show that this operation on $\widehat{G}$ makes $\widehat{G}$ into an abelian group. [Show that the identity is the map $g \mapsto 1$ for all $g \in G$ and the inverse of $\chi \in \widehat{G}$ is the map $g \mapsto \chi(g)^{-1}$.]*

(b) *If G is a finite abelian group, prove that $\widehat{G} \cong G$. [Write G as $\langle x_1 \rangle \times \cdots \times \langle x_r \rangle$ and if $n_j = |x_j|$ define χ_j to be the homomophism which sends x_i to $e^{2\pi i/n_j}$ and sends x_k to 1, for all $k \neq j$. Prove χ_j has order n_j in $\widehat{G}$ and $\widehat{G} = \langle \chi_1 \rangle \times \cdots \times \langle \chi_r \rangle$.]*

(This result is often phrased: a finite abelian group is self-dual. It implies that the lattice diagram of a finite abelian group is the same when it is turned upside down. Note however that there is no natural isomorphism between G and its dual (the isomorphism depends on a choice of a set of generators for G). This is frequently stated in the form: a finite abelian group is noncanonically isomorphic to its dual.)

Problem 17. *Let R be a ring with identity. Prove the following:*

(a) *Without assuming addition is commutative, show that it must follow from the other axioms.*

(b) $0a = a0 = 0$ *for all* $a \in R$.

(c) $(-a)b = a(-b) = -(ab)$ *for all* $a, b \in R$.

(d) $(-a)(-b) = ab$ *for all* $a, b \in R$.

(e) *the identity is unique in R and* $-a = (-1)a = a(-1)$ *for all* $a \in R$.

Problem 18. *Let p be prime and let R be a ring with identity of order p^2. Prove R is commutative.*

Problem 19. *Show that a non-zero ring R in which $x^2 = x$ for all $x \in R$ is of characteristic 2 and commutative.*

Problem 20. *Let R be a ring such that $x^3 = x$ for all $x \in R$. Show that R must be commutative.*