Recitation Problems
MAT 598
Week #2
January 30, 2015

Let M and N be smooth manifolds, and let $\pi_M : M \times N \to M$ and $\pi_N : M \times N \to N$ be the standard projection maps. Show that for any point $p = (p_M, p_N) \in M \times N$, the map

$$\alpha : T_p(M \times N) \to T_{p_M}M \oplus T_{p_N}N$$

defined by

$$\alpha(v) = (d(\pi_M)_p(v), d(\pi_N)_p(v))$$

is an isomorphism.

Note that $T_p(M \times N)$ and $T_{p_M}(M) \oplus T_{p_N}(N)$ have the same dimension. Therefore, if we show that α has a left inverse, then α must be an isomorphism. Also note that α is linear because $d(\pi_M)_p$ and $d(\pi_N)_p$ are.

Let $\iota_M : M \to M \times N$ be the map $m \mapsto (m, p_N)$ and $\iota_N : N \to M \times N$ be the map $n \mapsto (p_M, n)$. We now do a couple calculations:

$$\pi_M \circ \iota_M = \text{Id}_M,$$
$$\pi_N \circ \iota_N = \text{Id}_N,$$
$$\pi_M \circ \iota_N = p, \text{ and}$$
$$\pi_N \circ \iota_M = p.$$

From Proposition 3.6 we have

$$\text{Id}_{T_{p_M}M} = d(\pi_M \circ \iota_M)_{p_M} = d(\pi_M)_p \circ d(\iota_M)_{p_M},$$
$$\text{Id}_{T_{p_N}N} = d(\pi_N \circ \iota_N)_{p_N} = d(\pi_N)_p \circ d(\iota_N)_{p_N},$$
$$0 = d(\pi_M \circ \iota_N)_{p_N} = d(\pi_M)_p \circ d(\iota_N)_{p_N}, \text{ and}$$
$$0 = d(\pi_N \circ \iota_M)_{p_M} = d(\pi_N)_p \circ d(\iota_M)_{p_M}.$$

Now define a map:

$$\beta : T_{p_M} \oplus T_{p_N}M \to T_p(M \times N) \quad \text{by} \quad \beta(v_M, v_N) = d(\iota_M)_{p_M}(v_M) + d(\iota_N)_{p_N}(v_N).$$

Then for any $v_M \in T_{P_M}M$ and $v_N \in T_{p_N}N$, we have

$$\alpha \circ \beta(v_M, v_N) = \alpha\left(d(\iota_M)_{p_M}(v_M) + d(\iota_N)_{p_N}(v_N)\right) =$$

$$= \Big(d(\pi_M)_p(d(\iota_M)_{p_M}(v_M) + d(\iota_N)_{p_N}(v_N)), d(\pi_N)_p(d(\iota_M)_{p_M}(v_M) + d(\iota_N)_{p_N}(v_N)) \Big) =$$

$$\Big(d(\pi_M)_p \circ d(\iota_M)_{p_M}(v_M) + d(\pi_M)_p \circ d(\iota_N)_{p_N}(v_N),$$

$$d(\pi_N)_p \circ d(\iota_M)_{p_M}(v_M) + d(\pi_N)_p \circ d(\iota_N)_{p_N}(v_N) \Big) =$$

$$= \Big(\mathrm{Id}_{T_{p_M}}(v_M) + 0, 0 + \mathrm{Id}_{T_{p_N}}(v_N) \Big) = (v_M, v_N).$$

Therefore, $\alpha \circ \beta = \mathrm{Id}_{T_{p_M}(M) \oplus T_{p_N}(N)}$ and α must be an isomorphism. ■

Show that the map $q : \mathbb{S}^n \to \mathbb{RP}^n$ defined by

$$q(x^0, \dots, x^n) = [x^0 : \dots : x^n]$$

is a smooth covering map.

Let $p = [p^0 : \dots : p^n] \in \mathbb{RP}^n$. Choose $0 \leq i \leq n$ such that $p^i \neq 0$ and put

$$U_i = \{[x^0 : \dots : x^n] \in \mathbb{RP}^n : x^i \neq 0\}.$$

Then

$$q^{-1}(U_i) = \{(x^0, \dots, x^n) \in \mathbb{S}^n : x^i \neq 0\}.$$

Note that $q^{-1}(U_i)$ has two connected components: $U_i^+ = \{(x^0, \dots, x^n) \in \mathbb{S}^n : x^i > 0\}$ and $U_i^- = \{(x^0, \dots, x^n) \in \mathbb{S}^n : x^i < 0\}$. We must show that $q : U_i^{\pm} \to \mathbb{RP}^n$ is a diffeomorphism. We will do this for U_i^+; the proof for U_i^- is similar.

The fact that $q\big|_{U_i^+}$ is bijective is clear. Let $\psi : U_i \to \mathbb{R}^n$ be the standard chart on $\mathbb{RP}^n$. Define

$$\phi : U_i^+ \to \mathbb{R}^n \quad \text{by} \quad \phi(x^0, \dots, x^n) = \left(\frac{x^0}{x^i}, \dots, \widehat{\frac{x^i}{x^i}}, \dots, \frac{x^n}{x^i} \right),$$

where the "hat" notation indicates that we are removing that coordinate. Then each component function of ϕ is smooth because $x^i \neq 0$. Since each component function of ϕ is smooth then ϕ is smooth.

Now define $\phi^{-1} : \mathbb{R}^n \to U_i^+$ by

$$\phi^{-1}(x^1, \dots, x^n) =$$

$$\left(\frac{x^1}{\sqrt{1 + (x^1)^2 + \dots + (x^n)^2}}, \dots, \frac{1}{\sqrt{1 + (x^1)^2 + \dots + (x^n)^2}}, \dots, \frac{x^n}{\sqrt{1 + (x^1)^2 + \dots + (x^n)^2}} \right).$$

It's straightforward to show that ϕ^{-1} is actually the inverse of ϕ. It is clear that each component of ϕ^{-1} is smooth because the denominator is never 0. Since each component is smooth, then ϕ^{-1} is smooth. It follows that (U_i^+, ϕ) is a smooth chart of $\mathbb{S}^1$. In order to show that $q\big|_{U_i^+}$ is a diffeomorphism, we must show that

$$\phi^{-1} \circ q \circ \psi : \mathbb{R}^n \to \mathbb{R}^n$$

is smooth with smooth inverse. Using the same calculation as show that ϕ^{-1} is the inverse of ϕ, one can easily see that $\phi^{-1} \circ q \circ \psi = \mathrm{Id}_{\mathbb{R}^n}$, which is smooth with smooth inverse. ∎