Extra Problems

MAT543

Fall 2012

December 3, 2012

These are all the extra problems I posted in Fall 2012.

Problem 1. Prove or disprove the following: *Let H_1, H_2 be groups and define $G = H_1 \times H_2$. Then every subgroup of G is of the form $K_1 \times K_2$ where $K_1 \leq H_1$, $K_2 \leq H_2$.*

Problem 2. *Let G be a group of order n. Show that G is cyclic if and only if G has a unique subgroup of order d for all positive $d \mid n$.*

Problem 3. *Let $Z(G)$ be the center of the group G. Prove if $G/Z(G)$ is cyclic, then G is abelian. In other words, $G/Z(G) = \{e\}$.*

Problem 4. *Consider the additive group $\mathbb{Q}/\mathbb{Z}$. Prove:*

(a) For each $n \in \mathbb{N}$ there is an element of $\mathbb{Q}/\mathbb{Z}$ of order n.

(b) There is a unique subgroup of $\mathbb{Q}/\mathbb{Z}$ of order n for each $n \in \mathbb{N}$.

Problem 5. *Let G be a group and let M be a maximal subgroup of G. That is, M is a proper subgroup of G and for any subgroup H of G, if $M \subsetneq H$, then $H = G$. Prove that if M is normal in G and $|G : M| < \infty$, then $|G : M|$ is prime.*

Problem 6. *Let G be a group and let N be a normal subgroup of index n. Show that $g^n \in N$ for all $g \in G$.*

Problem 7. *(a) Prove that the additive group of the rational numbers is not cyclic.*

(b) Prove that a finitely generated subgroup of the rational numbers is cyclic.

Problem 8. *Let G be a group and let N be the subgroup of G generated by $\langle x^{-1}y^{-1}xy : x, y \in G\rangle$. Show that N is normal in G and G/N is abelian.*

Problem 9. *Show that S_n can be generated by the following sets.*

1. *The set of transpositions.*
2. *The set of cycles.*
3. *The set $\{(1\ 2), (1\ 2\ 3 \cdots n)\}$.*

Problem 10. *Let G be a subgroup of S_n. Show that if G contains an odd permutation, then $G \cap A_n$ has index 2 in G.*

Problem 11. *Show that if G is a subgroup of S_n of index 2, then $G = A_n$.*

Problem 12. *Let $n \geq 5$. Show that the only proper, nontrivial normal subgroup of S_n is A_n.*

Problem 13. *Let F_1 be the free group on one generator. Show that $F_1 \times F_1$ is not a free group. (Hint: If it was, what would the generators be?)*

Problem 14. *Let F be the free group on the set $\{x, y\}$. Let $\alpha : F \to \mathbb{Z}/2\mathbb{Z}$ be a homomorphism such that $\alpha(x) = 1 + \mathbb{Z}/2\mathbb{Z} = \alpha(y)$. Explain why we know that α defines a (unique) homomorphism on F. Find a minimal set of generators for the kernel of α. Is the kernel of α a free group?*

Problem 15. *Let G, H be groups. Show that if $G \rtimes \{e_H\}$ and $\{e_G\} \rtimes H$ are both normal subgroups of $G \rtimes H$, then $G \rtimes H \cong G \times H$. (Hint: Show that if the hypothesis holds, then the homomorphism underlying in the semi-direct product is the identity homomorphism)*

Problem 16. *Let p be an odd prime. Show that $D_p \cong P \rtimes Q$ where $P \in Syl_p(D_p)$ and $Q \in Syl_2(D_p)$. Let r correspond to the "rotation" of the p-gon and let s correspond the the "reflection." Show that the operation on $P \rtimes Q$ corresponds to the operation of D_p. That is, $sr = r^{-1}s$ in D_p so we must have $(r, e) \bullet (e, s) = (e, s) \bullet (e, r^{-1})$ in $P \rtimes Q$.*

Problem 17. *Show that there is a nontrivial semi-direct product of C_4 and S_3. That is, show that there is a way to define an operation of $C_4 \rtimes S_3$ such that $C_4 \rtimes S_3 \neq C_4 \times S_3$. Another way to state the problem would be: show that we can define an operation on the set of ordered pairs (c, s) with $c \in C_4$ and $s \in S_3$ so that $C_4 \trianglelefteq C_4 \rtimes S_3$, but $S_3 \ntrianglelefteq C_4 \rtimes S_3$.*

Problem 18. *Let G be a finite simple group and let $k \in \mathbb{N}$ be the smallest positive integer such that $|G| \mid k!$. Prove that if H is a proper subgroup of G, then $|G : H| \geq k$.*

Problem 19. *Let G be an infinite simple group. If $H \leq G$, then $|G : H|$ is infinite as well.*

Problem 20. *Let G be a finite simple group with $|G| \geq 60$. Prove that G has no subgroups of index less than 5.*

Problem 21. *Let G be a finite simple group with a subgroup H of prime index p. Show that p must be the largest prime dividing the order of G.*

Problem 22. *Let G be a finite group with order p^k for some prime p and some $k \in \mathbb{N}$. Show that $Z(G)$ is nontrivial (i.e. the center of G contains a non-identity element).*

Problem 23. *Let G be a group of order p^2 for p a prime. Show that G is abelian.*

Problem 24. *Let G be a group of order 160. Show that G is not simple. Furthermore, show that G must have a normal 2-subgroup.*

Problem 25. *Suppose G is a simple group of order $168 = 2^3 \cdot 3 \cdot 7$.*

(a) *Prove G is not abelian.*

(b) *Determine the number of elements of order 7 in G.*

(c) *Prove that G has no proper subgroups of index less than 7 (hint: use the homomorphism to S_m given by a subgroup of index m).*

(d) *If H_2 is a Sylow 2-subgroup of G and H_7 is a Sylow 7-subgroup of G, then G is generated by $H_2 \cup H_7$.*

(e) *Prove that the number of Sylow 3-subgroups in G is a multiple of 7.*

Problem 26. *Let G be a group of order 36. Show that G has a normal subgroup of order 3 or 9.*

Problem 27. *Let G be a group of order pqr, where $p > q > r$ are primes. Let P be a Sylow p-subgroup of G and assume P is not normal in G. Show that a Sylow q-subgroup of G must be normal.*

Problem 28. *Let G be a group of order $231 = 3 \cdot 7 \cdot 11$. Show that G has normal subgroups of orders 7 and 11. Furthermore, show that the normal subgroup of order 11 is contained in the center.*

Problem 29. *Let R be a ring with identity, 1_R. Prove the following:*

(a) *Without assuming addition is commutative, show that it must follow from the other axioms.*

(b) $0_R a = a 0_R = 0_R$ *for all $a \in R$.*

(c) $(-a)b = a(-b) = -(ab)$ *for all $a, b \in R$.*

(d) $(-a)(-b) = ab$ *for all $a, b \in R$.*

(e) *the identity is unique in R and $-a = (-1_R)a = a(-1_R)$ for all $a \in R$.*

Problem 30. *Show that a non-zero ring R in which $x^2 = x$ for all $x \in R$ is of characteristic 2 and commutative.*

Problem 31. *Let R be a ring such that $x^3 = x$ for all $x \in R$. Show that R must be commutative.*

Problem 32. *Show that a finite commutative ring with no zero divisors is a field.*

Problem 33. *Let R be a finite ring with identity.*

(a) *Show that if R has order p, then R is commutative.*

(b) *Show that if R has order p, then R is a field.*

(c) *Show that if R has order p^2, then R is commutative.*

Problem 34. *Let R be a subring of a field F such that for all $x \in F$, either $x \in R$ or $x^{-1} \in R$. Let I and J be two ideals of R. Show that $I \subseteq J$ or $J \subseteq I$.*

Problem 35. *Let R be a nonzero commutative ring with 1. Show that if I is an ideal of R such that $1 + a$ is a unit in R for all $a \in I$, then I is contained in every maximal ideal of R.*

Problem 36. *Let R be a commutative ring with identity and let $x \in R$ be a non-nilpotent element. Show that there exists a prime ideal of R that does not contain x.*

Problem 37. *Let R be a commutative ring with 1 in which every ideal is prime. Show that R is a field.*

Problem 38. *Let R be a commutative ring with identity. Show that the set of all nilpotent elements of R is an ideal of R.*

Problem 39. *A commutative ring R is said to be Noetherian if $\mathfrak{a}_1 \subseteq \mathfrak{a}_2 \subseteq \mathfrak{a}_3 \subseteq \dots$ is an ascending chain of ideals of R, then there exists $n \in \mathbb{N}$ such that $\mathfrak{a}_m = \mathfrak{a}_n$ for all $m \geq n$. Show that every ideal of a Noetherian ring is finitely generated.*

Problem 40. *Let p be a prime number and let R be the set of all rational numbers with denominator prime to p. R is a subring of $\mathbb{Q}$.*

(a) What are the units of R?

(b) Show that R is a principle ideal domain.

(c) Show that R has a unique maximal ideal M, find a generator for M, and identify R/M.

Problem 41. *Let R be a commutative ring with identity.*

(a) Show that every maximal ideal is prime without using quotient rings.

(b) If R is a PID, show that every prime ideal is maximal with using quotient rings.

(c) Give an example of a ring R with identity and an ideal $I \subseteq R$ such that I is prime, but not maximal.

Problem 42. *Let $R = \mathbb{Z}[\sqrt{-3}]$ and $S = \mathbb{Z}[i]$. Show that there is no homomorphism $\phi : R \to S$ such that $\phi(1_r) = 1_S$.*

Problem 43. *Let D be a unique factorization domain and F its field of fractions. Prove that if d is an irreducible element in D, then there is not $x \in F$ such that $x^2 = d$.*

Problem 44. *Let D be a unique factorization domain and p a fixed irreducible element of D such that if q is any irreducible element of D, the q is an associate of p. Show the following:*

(a) If d is a nonzero element of D, then d is uniquely expressible in the form up^n, where y is a unit of D and n is a nonnegative integer.

(b) D is a Euclidean domain.