Recitation Problems
MAT 598
Week #1
January 23, 2015

Let $F : M \longrightarrow N$ be a continuous function and let $C^\infty(M)$ denote the algebra of smooth functions from a manifold M to $\mathbb{R}$. Then F defines a linear map $F^* : C^\infty(N) \longrightarrow C^\infty(M)$ by $F^*(g) = g \circ F$. Show that F is smooth if and only if $F^*(C^\infty(N)) \subseteq C^\infty(M)$.

($\Rightarrow$) If F is smooth, then for all $g \in C^\infty(N)$ we know that $g \circ F = F^*(g) \in C^\infty(M)$. ✓

($\Leftarrow$) Suppose $F^*(C^\infty(N)) \subseteq C^\infty(M)$. Fix $p \in M$ and let (U, ϕ) and (V', ψ) be charts at p and $F(p)$, respectively. Let $V \subseteq V'$ be open such that $\overline{V} \subseteq V'$. Since V is open and F is continuous, the $U \cap F^{-1}(V)$ is open in M and it suffices (by Proposition 2.5) to show that the map

$$\psi \circ F \circ \phi^{-1} : \phi(U \cap F^{-1}(V)) \longrightarrow \psi(V)$$

is smooth.

Let $\psi^i : V \longrightarrow \mathbb{R}$ denote the i-th coordinate function of ψ. By Lemma 2.26, there is a smooth function $f^i \in C^\infty(N)$ such that $f^i\big|_V = \psi^i$. Then we have

$$\left(\psi \circ F \circ \phi^{-1}\right)^i = \psi^i \circ F \circ \phi^{-1} = f^i\big|_V \circ F \circ \phi^{-1}.$$

Note that $F \circ \phi^{-1}(\phi(U \cap F^{-1}(V)) \subseteq V$, which implies that

$$f^i\big|_V \circ F \circ \phi^{-1} = f^i \circ F \circ \phi^{-1}.$$

By assumption, $f^i \circ F = F^*(f^i) \in C^\infty(M)$. Also, ϕ^{-1} is smooth (see below). Therefore,

$$\left(\psi \circ F \circ \phi^{-1}\right)^i = (f^i \circ F) \circ \phi^{-1} : \phi(U \cap F^{-1}(V)) \longrightarrow \psi(V)$$

is smooth. Since i was arbitrary, then each component function of $\psi \circ F \circ \phi^{-1}$ is smooth, whence $\psi \circ F \circ \phi^{-1}$ is smooth. ✓ ■

Let $M_{m,n}(R)$ denote the set of all $m \times n$ matrices with entries in $\mathbb{R}$. Show that the following subsets of $M_{m,n}(\mathbb{R})$ are smooth manifolds:

(i) $GL_n(\mathbb{R})$ if $m = n$.

(ii) $SL_n(\mathbb{R})$ if $m = n$.

(iii) The set of matrices with rank k for any $0 \leq k \leq \min(m, n)$.

(i) If $m = n$, then the determinant (which we will call det) is well-defined. Then

$$GL_n(\mathbb{R}) = \overset{-1}{\det}(\mathbb{R} \setminus \{0\}).$$

Therefore, since det is continuous, $GL_n(\mathbb{R})$ is an open subset of a smooth manifold. Hence $GL_n(\mathbb{R})$ is a smooth manifold.

(ii) Again, we are assuming $m = n$. Note

$$SL_n(\mathbb{R}) = \overset{-1}{\det}(\{1\}).$$

By viewing $M_{n,n}(\mathbb{R})$ as R^{n^2}, we see that $SL_n(\mathbb{R})$ is a level set of det. Therefore, $SL_n(\mathbb{R})$ is a smooth manifold (See Example 1.32).

(iii) A matrix with rank k has a $k \times k$ minor with non-zero determinant (you can see this by row-reducing). The rest of the argument is the same as (i).

■

We had a small discussion about chart maps being smooth. By definition, the transition maps are smooth (they are diffeomorphisms), but there is no mention as to whether the chart maps themselves are smooth. They are, and here is why.

Let M be a smooth manifold. Recall (Lee, page 32) that a map from $F : M \longrightarrow \mathbb{R}^n$ is smooth if and only if for all $p \in M$ there exists a smooth chart (U, ϕ) with $p \in U$ such that $f \circ \phi^{-1} : \phi(U) \longrightarrow \mathbb{R}^n$ is smooth.

Let (U, ϕ) be any (smooth) chart on M. Then U is open and therefore a smooth manifold in its own right. Furthermore, (U, ϕ) is still a chart on U. Then for any $p \in U$, the map

$$\mathrm{Id} = \phi \circ \phi^{-1} : \phi(U) \longrightarrow \mathbb{R}^n$$

is smooth. Therefore, ϕ is smooth at every point of U. ■